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THE NONSUBSTITUTION THEOREM WITH JOINT PRODUCTION

by
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1. *Introduction*

One of the most striking results of fixed coefficient technology is the nonsubstitution theorem. The theorem shows that under certain conditions there is a *unique* set of techniques of production which will be used regardless of the structure of demand. Consequently, it is unnecessary to have any information about consumer demand in order to calculate the equilibrium prices. Some authors have claimed that the result no longer holds if joint production is allowed (see, for example, Johansen, 1972; Koopmans, 1953; Otani, 1974).

We state and prove the theorem and give examples where all the assumptions are satisfied and joint production is also present.

2. *The Model*

In the economy, there is a set of m industries which produce n (≥ 2) goods. There is a single non-producible factor of production (which we call *labor*). Each industry is completely specified by a set of (input-output) coefficients. Industry i is described by

$$(a_{i1}, \ldots, a_{in}, b_{in+1}) \qquad i = 1, \ldots, m$$

where a_{ij} = the *net* amount of good j produced at the unit level of operation of industry i.

We assume constant returns to scale (and no externality) for every

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industry. So we can *normalize* industry i by setting $b_{in=1} = 1$ for $i = 1, \ldots, m$.

Notations: By A we will denote the $m \times n$ matrix whose ijth element is a_{ij}. The ith row of A will be denoted by $a_{i\cdot}$.

Let x be a (row) vector in R^m. Then $x \gg 0$ will mean $x_i > 0$ for $i = 1, \ldots, m$, and $x = (x_1, \ldots, x_m)$, and $x \geq 0$ will mean $x_i \geq 0$ for $i = 1, \ldots, m$.

Let $R^m_+ = \{x : x \geq 0\}$. Define $\partial(S)$ to be the *boundary* of the set $S \subseteq R^n$; let S^c denote the *complement* of the set S in R^n.

If A is an $n \times n$ matrix and its inverse exists, we denote the inverse by A^{-1}.

Assumptions:

(A1) For all $x \in R^m_+$, $xA \geq 0$ implies that $\{i : x_i > 0\}$ has at least n elements.

Remark: (A1) means that to produce all goods in the economy at least n industries are needed. However, (A1) is compatible with joint production (for $n \geq 3$) as the following example shows:

Example: $a_1 = (1, 1, -1)$ $a_2 = (1, -1, 1)$ $a_3 = (-1, 1, 1)$. Any two of the above cannot produce all the three goods but the three together can.

(A2) There exist indices $i_1, \ldots, i_k$ such that $\sum_{t=1}^{k} x_{i_t} a_{i_t} \gg 0$, $k \geq n$;

where $x_{i_t} > 0$ for $t = 1, \ldots, k$. (A2) means that there exists a set of n industries which can realize a strictly positive net output vector.

3. *Preliminaries*

Consider a set of n rows from A. Denote them by $\{a_1, \ldots, a_n\}$ without any loss of generality.

Let $D = \left\{d \in R^n : d \gg 0;\right.$ there exists x in R^n_+ such that $x \gg 0$ and $\left.\sum_{i=1}^{n} x_i a_i = d\right\}$

Lemma 1: Let (A1) hold. Let $D \neq \emptyset$ for a set of industries $1, \ldots, n$ as defined above. Then $\{a_1, \ldots, a_n\}$ is a set of linearly independent vectors in R^n.

Proof: Suppose not. Consider the following Linear Programming problem: minimize $\sum_{i=1}^{n} x_i$ subject to $\sum_{i=1}^{n} x_i a_i = d$; where $x_i \geq 0$ for $i = 1, \ldots, n$ and where $d \in D$. Since $D \neq \emptyset$, the solution space is nonempty and the solutions are bounded below by 0 vector. Let S denote the set of solutions to the above problem. Then S is a nonempty compact subset of R^n. So, by Weirstrass's Theorem, such a minimum exists. By supposition, we can find a subset $\{a_{j_1}, \ldots, a_{j_t}\}$ with $t < n$ of $\{a_1, \ldots, a_n\}$ such that it is a basis of the non-negative solution of $\sum_{i=1}^{n} x_i a_i = d$ (follows from a well known Theorem in Linear Programming, see Gale, 1960, Theorem 2.11).

Let the solution be denoted by $\{x^*_{j_1}, \ldots, x^*_{j_t}\}$, $t < n$. Define $x^*_i = 0$ for $i \in \{\{1, \ldots, n\}/\{j_1, \ldots, j_t\}\}$. Then $\sum_{i=1}^{n} x^*_i a_i = d$ with $d \gg 0$. But clearly $x^*_i \not\gg 0$ for every $i = 1, \ldots, n$. This contradicts (A1) Q.E.D.

Lemma 2: Assume (A1) holds. If $D \neq \emptyset$ then D is an open subset of R^n_+ (in the usual topology).

Proof: Suppose not. Then there exists $d^* \in D$ such that every open set containing d^* contains a point of D^c. Hence there exists a sequence $\{d^q\}$ with $d^q \in D^c$ for every $q \geq 1$ and $d^q \to d^*$. Without loss of generality we can truncate $\{d^q\}$ suitably to get $d^q \gg 0$ for every $q \geq 1$. By lemma 1, there exists a *unique* (row) vector $x^q = (x^q_1, \ldots, x^q_n)$ such that $d^q = \sum_{i=1}^{n} x^q_i a_i$ for every $q \geq 1$ (since $\{a_1, \ldots, a_n\}$ are linearly independent).

By assumption, for every $q \geq 1$, $d^q \notin D$ and $d^q \gg 0$. So for a fixed q there does *not* exist $x^q \downarrow 0$ such that $d^q = \sum_{i=1}^{n} x^q_i a_i$. Hence there exists $x^q_i \leq 0$ for some i in $\{1, \ldots, n\}$ (for fixed q). Thus, there exists i_0 such that $x^q_{i_0} \leq 0$ for infinitely many q's. So, there is a subsequence $\{q_m\}$ of the sequence $\{q\}$ satisfying:

$$x^{q_m}_{i_0} \leq 0 \text{ for all } m \geq 1 \tag{3.1}$$

On the other hand, $d \in D$. Thus there exists $x^0 \gg 0$ such that $d^0 = \sum_{i=1}^{n} x^0_i a_i$ and $x^0_{i_0} > 0$ where $x^0 = (x^0_1, \ldots, x^0_n)$. But this contradicts (3.1) Q.E.D.

Lemma 3: Assume (A1) holds and $D \neq \emptyset$. If $d \geq 0$ and $d = \sum_{i=1}^{n} x_i a_i$ where $x_i \geq 0$, then $d \in \bar{D}$ (where $\bar{D}$ is the closure of D in the usual topology).

Proof: Suppose $d \notin D$. Let $d_0 \in D$. Define: $d_e = e\,d + (1-e)\,d_0$ for $0 \leq e \leq 1$. Since $\bar{D}$ is closed, there exists $e \in (0,1)$ such that $d_e \notin D^c$. Let $e^* = \sup\{e : d_e \in \bar{D}\}$. Since $d \notin \bar{D}$, it follows that $e^* < 1$. Moreover $d_e^* \in \bar{D}$. But, by definition, $d_e^* \in \partial(\bar{D})$. Clearly, $e^* < 1$ implies that $d_e^* \gg 0$. Let $d_0 = \sum_{i=1}^{n} x_i^0 a_i$ where $x_i^0 > 0$ for all $i = 1, \ldots, n$.
Then,

$$d_e^* = e^* d + (1-e^*)\,d_0 = e^* \sum_{i=1}^{n} x_i a_i + (1-e^*) \sum_{i=1}^{n} x_i^0 a_i$$

$$= \sum_{i=1}^{n} \{e^* x_i + (1-e^*)\,x_i^0\}\,a_i = \sum_{i=1}^{n} y_i a_i$$

where $$y_i = e^* x_i + (1-e^*)\,x_i^0$$

Since $e^* < 1$ and $x_i \geq 0$ for $i = 1, \ldots, n$, it follows that $y_i > 0$ for $i = 1, \ldots, n$. Thus $d_e^* \in D$. But then by lemma 2, there exists an open set Z containing d_e^* with $Z \subseteq D$. This contradicts the fact that $d_e^* \in \partial(\bar{D})$. Hence $d \in \bar{D}$. Q.E.D.

Corollary: Let $\{\bar{a}_1, \ldots, \bar{a}_n\}$ be n rows of A such that $\sum_{i=1}^{n} x_i \bar{a}_i \gg 0$ for some $x = (x_1, \ldots, x_n) \gg 0$. Then the matrix $\bar{A} = \{\bar{a}_1, \ldots, \bar{a}_n\}$ is invertible and $(\bar{A})^{-1} \geq 0$.

Proof: In Lemma 3, set $e_1 = (1, 0, \ldots, 0)$; $e_2 = (0, 1, 0, \ldots, 0) \ldots e_n = (0, 0, \ldots, 0, 1)$. Then there exists $x_1, \ldots, x_n \geq 0$ such that $x_1 \bar{A} = e_1, \ldots, x_n \bar{A} = e_n$. Thus, $\bar{D} = R_+^n$. Hence $(\bar{A})^{-1}$ exists. Since $(\bar{A})^{-1} e_i = x_i \geq 0$ it follows that $(\bar{A})^{-1} \geq 0$.

4. *The Nonsubstitution Theorem*

By the convention adopted earlier, when a set of activities $\{a_1, \ldots, a_k\}$ are operated at the levels $\{x_1, \ldots, x_k\}$ the total labor requirement for this operation is $\sum_{i=1}^{k} x_i$. We have to show that there exist n rows of A which are capable of realizing *any* non-negative vector of outputs by involving the "least use" of labor.

Theorem 1: Assume that (A1) and (A2) hold. Then there exist n linearly independent rows of A, $\{a_1, \ldots, a_n\}$ such that for all $c \geq 0$, there exists $x = (x_1, \ldots, x_n)$ with $\sum_{i=1}^{n} x_i a_i = c$. Moreover, if there exists a set $\{s_1, \ldots, s_k\}$ such that $(x_{s_1}, \ldots, x_{s_k}) \geq 0$ and $\sum_{j=1}^{k} x_{s_j} a_{s_j} = c \geq 0$ then $\sum_{j=1}^{k} x_{s_j} \geq \sum_{i=1}^{n} x^i$.

Proof: Let $c = (c_1, \ldots, c_n) \gg 0$. Consider the following Linear Programming problem:

(P) minimize $\sum_{i=1}^{m} x_i^0$ subject to $\sum_{i=1}^{m} x_i^0 a_i = c$

By (A2), (P) has a feasible solution. Let $\{a_1, \ldots, a_k\}$ be the optimal basis. Then $k \leq n$. But, by (A1), $k > n - 1$. Therefore $k = n$. We shall show that this set satisfies all the properties asserted in the statement of the theorem.

We first show that $\{a_1, \ldots, a_n\}$ minimizes the cost of producing e_k where 1 at the kth place and 0 elsewhere for $k = 1, \ldots, n$. By lemma 3, there are (unique) vectors $x^k = (x_1^k, \ldots, x_n^k) \geq 0$ $k = 1, \ldots, n$ such that

$$\sum_{t=1}^{n} x_t^k a_t = c_k e_k \qquad k = 1, \ldots, n \tag{4.1}$$

Therefore

$$\sum_{k=1}^{n} \sum_{t=1}^{n} x_t^k a_t = \sum_{k=1}^{n} c_k e_k$$

so

$$\sum_{t=1}^{n} \tilde{x}_t a_t = c$$

where

$$\tilde{x}_t = \sum_{k=1}^{n} x_t^k \geq 0$$

But since $\{a_1, \ldots, a_n\}$ is an optimal basis, $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is a unique, cost minimizing vector.

Suppose now $\{a_1, ..., a_n\}$ does not minimize the labor cost for at least one k. Without loss of generality, suppose the k satisfying the condition of the preceding sentence is n. Then, there is a set of vectors $\{\bar{a}_1, ..., \bar{a}_{k_n}\}$ of rows of A with $k_n \leq m$ and numbers $\{\bar{x}_1^n, ..., \bar{x}_{k_n}^n\}$ such that

$$\sum_{t=1}^{k_n} \bar{x}_t^n \bar{a}_t \geq c_n e_n \tag{4.2}$$

and

$$\sum_{t=1}^{k_n} \bar{x}_t^n < \sum_{t=1}^{n} x_t^n \tag{4.3}$$

By combining (4.1) and (4.2) we get

$$\sum_{t=1}^{k_n} \bar{x}_t^n \bar{a}_t + \sum_{k=1}^{n-1} \sum_{t=1}^{n} x_t^k a_t \geq c_n e_n + \sum_{k=1}^{n-1} c_k e_k = c$$

and from (4.3) we get

$$\sum_{t=1}^{k_n} \bar{x}_t^n + \sum_{j=1}^{n-1} \sum_{t=1}^{n} x_t^j < \sum_{t=1}^{n} x_t^n + \sum_{j=1}^{n-1} \sum_{t=1}^{n} x_t^j = \sum_{t=1}^{n} \tilde{x}_t$$

But this contradicts the fact that $\{\tilde{x}_1, ..., \tilde{x}_n\}$ is an optimal solution of (P), since there is another feasible vector which entails a lower labor cost. This proves the theorem. Q.E.D.

Remark: The method of proof of Theorem 1 is due to Chunder (1974). However, he dealt with the no-joint-production case.

Theorem 2: In the above formulation there exists a unique $p \gg 0$ associated with $\{a_1, ..., a_n\}$ and it is invariant with respect to changes in the composition of the final demand vector.

Proof: Let p_i be the price per unit of net product i associated with the set of activities $\{a_1, ..., a_n\}$ when the rate of profit is zero and the exogenously specified price per unit of labor is unity. Then by definition it follows that $\bar{A}p = e$ where $\bar{A}$ is the matrix formed by $\{a_1, ..., a_n\}$, p is a *column* vector in R^n; and e is the (column) vector in R^n with 1 in every row. By corollary to Lemma 3, $(\bar{A})^{-1} \geq 0$. Hence $p = (\bar{A})^{-1} e \geq 0$. Suppose $p_1 = 0$. Then $x^1 \bar{A} = c_1 e_1$ where $x^1 = (x_1^1, ..., x_n^1)$ is as defined in (4.1). Note that $x^1 \neq 0$ but $x^1 \bar{A}p = c_1 e_1 = c_1 p_1 = 0$ $\bar{A}p = e$ and $x^1 \neq 0$. So, $0 = c_1 p_1 = x^1 \bar{A} p \neq 0$ a contradiction! Hence $p \gg 0$. Q.E.D.

Remark: In the example given in section 2, if we take p to be the *column* vector with unity in every row and $\bar{A} = \{a_1, a_2, a_3\}$ then all the assumptions hold. Thus the discussion of Theorem 2 is not vacuous.

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IL TEOREMA DI NON SOSTITUZIONE CON PRODUZIONE CONGIUNTA

È noto che nella tecnologia di Leontief senza produzione congiunta il teorema di non sostituzione è valido. Alcuni autori hanno affermato che non lo è se si introduce la produzione congiunta. Qui viene dimostrato il teorema di non sostituzione sotto condizioni che ammettono certi tipi di produzione congiunta.